

CYCLICITY IN THE HARMONIC DIRICHLET SPACE

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ABSTRACT. The harmonic Dirichlet space is the Hilbert space of functions $f \in L^2(\mathbb{T})$ such that

$$\|f\|_{\mathcal{D}(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} (1 + |n|) |\widehat{f}(n)|^2 < \infty.$$

We give sufficient conditions for f to be cyclic in $\mathcal{D}(\mathbb{T})$, in other words, for $\{\zeta^n f(\zeta) : n \geq 0\}$ to span a dense subspace of $\mathcal{D}(\mathbb{T})$.

1. INTRODUCTION

Let \mathcal{X} be a topological linear space of complex functions on the unit circle \mathbb{T} such that the shift operator S , given by

$$S(f)(\zeta) := \zeta f(\zeta), \quad f \in \mathcal{X},$$

is an isomorphism of \mathcal{X} onto itself.

A closed subspace \mathcal{M} of \mathcal{X} is called *invariant* if $S(\mathcal{M}) \subset \mathcal{M}$. It is said to be *1-invariant* (or *simply invariant*) for S if $S(\mathcal{M}) \subsetneq \mathcal{M}$, and it is called *2-invariant* (or *doubly invariant*) if $S(\mathcal{M}) = \mathcal{M}$. The latter condition is equivalent to the invariance of \mathcal{M} under multiplication by both ζ and $\bar{\zeta}$.

Let \mathbb{Z} denote the integers and let $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$. Given $f \in \mathcal{X}$, we write

$$\begin{aligned} [f]_{\mathbb{N}} &:= \overline{\text{Span}}^{\mathcal{X}} \{z^n f : n \in \mathbb{N}\}, \\ [f]_{\mathbb{Z}} &:= \overline{\text{Span}}^{\mathcal{X}} \{z^n f : n \in \mathbb{Z}\}. \end{aligned}$$

A function f is said to be *1-invariant* if the space $[f]_{\mathbb{N}}$ is 1-invariant for S . We say that a function $f \in \mathcal{X}$ is *cyclic* (resp. *bicyclic*) for \mathcal{X} if $[f]_{\mathbb{N}} = \mathcal{X}$ (resp. $[f]_{\mathbb{Z}} = \mathcal{X}$).

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Let us begin with the classical case, namely $\mathcal{X} = L^2(\mathbb{T})$. By a well-known theorem of Wiener, the 2-invariant subspaces have the form

$$\mathcal{M} = \{f \in L^2(\mathbb{T}) : f = 0 \text{ a.e. on } \mathbb{T} \setminus \sigma\},$$

where σ is a Borel subset of \mathbb{T} (see e.g. [11, p.8, Theorem 1.2.1]). It follows from Szegő's infimum theorem that a function f is 1-invariant in $L^2(\mathbb{T})$ if and only if $\log |f| \in L^1(\mathbb{T})$ (see e.g. [10, p.12, Corollary 4]).

For $\mathcal{X} = \mathcal{C}^\infty(\mathbb{T})$, Makarov [7] gave a complete description of the invariant subspaces of S . He also obtained the following characterization of 1-invariant functions of $\mathcal{C}^\infty(\mathbb{T})$.

Theorem (Makarov [7, p.3]). *A function f is 1-invariant in $\mathcal{C}^\infty(\mathbb{T})$ if and only if $\log |f| \in L^1(\mathbb{T})$.*

The case $\mathcal{X} = \mathcal{C}^n(\mathbb{T})$ ($n \geq 1$) is more complicated, and no characterization of 1-invariant functions is known ([7, p.3], see also [8, Theorem 1.3] or [9, Theorem 5]).

We shall focus our attention on the *harmonic Dirichlet space* $\mathcal{D}(\mathbb{T})$. This is the set of functions $f \in L^2(\mathbb{T})$ whose Fourier coefficients satisfy

$$\mathcal{D}(f) := \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 |n| < \infty.$$

It becomes a Hilbert space if endowed with the norm $\|\cdot\|_{\mathcal{D}(\mathbb{T})}$, given by

$$\|f\|_{\mathcal{D}(\mathbb{T})}^2 := \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(f) = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 (1 + |n|).$$

According to Douglas' formula [3], we have

$$\mathcal{D}(f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

This can also be written

$$\mathcal{D}(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \mathcal{D}_\zeta(f) |d\zeta|,$$

where $\mathcal{D}_\zeta(f)$ is the so-called local Dirichlet integral of f at ζ , given by

$$\mathcal{D}_\zeta(f) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'|, \quad \zeta \in \mathbb{T}.$$

As $\mathcal{D}(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, if a function f is cyclic for $\mathcal{D}(\mathbb{T})$, then it is also cyclic for $L^2(\mathbb{T})$. So by Szegő's theorem we have

$$f \text{ is cyclic for } \mathcal{D}(\mathbb{T}) \Rightarrow \int_{\mathbb{T}} \log |f(\zeta)| |d\zeta| = -\infty.$$

In [12], Ross, Richter and Sundberg gave a complete characterization of the 2-invariant subspaces \mathcal{M} of $\mathcal{D}(\mathbb{T})$ in terms of their zero

sets. In order to state their result, we need to introduce the notion of logarithmic capacity (see for instance [6, §2.4]).

The *energy* of a Borel probability measure μ on \mathbb{T} is defined by

$$I(\mu) := \iint_{\mathbb{T}^2} \log \frac{1}{|\zeta - \zeta'|} d\mu(\zeta) d\mu(\zeta') = \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n}.$$

Then we define the *logarithmic capacity* of a Borel subset E of \mathbb{T} by

$$c(E) := 1/\inf\{I(\mu) : \mu \in \mathcal{P}(E)\},$$

where $\mathcal{P}(E)$ denotes the set of probability measures supported on a compact subset of E . We say that a property holds *quasi-everywhere* (q.e.) if it holds everywhere outside a set of logarithmic capacity zero.

It is known that, if $f \in \mathcal{D}(\mathbb{T})$, then the radial limit of the Poisson integral of f exists q.e. and is equal to f a.e. (for more details we refer to [12] and the references therein). In the sequel, f will denote this limit, and will therefore be defined q.e. on \mathbb{T} . We shall write $\mathcal{Z}(f)$ for the zero set of f , namely

$$\mathcal{Z}(f) := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}.$$

Note that this set is defined up to sets of logarithmic capacity zero.

Theorem (Richter–Ross–Sundberg [12]). *\mathcal{M} is a 2-invariant subspace of $\mathcal{D}(\mathbb{T})$ if and only if there exists a measurable set $E \subset \mathbb{T}$ such that*

$$\mathcal{M} = \mathcal{D}_E := \{f \in \mathcal{D}(\mathbb{T}) : f|_E = 0 \text{ q.e.}\}.$$

Note that the problem of characterization of 1-invariant subspaces of $\mathcal{D}(\mathbb{T})$ remains open. It was proved in [1, 13] that, for each $n \in \mathbb{N} \cup \{\infty\}$, there exists an invariant subspace \mathcal{M} of $\mathcal{D}(\mathbb{T})$ such that $\dim(\mathcal{M}/S(\mathcal{M})) = n$. This suggests that the lattice of 1-invariant subspaces has a very complicated structure.

As a direct consequence of the Richter–Ross–Sundberg theorem, we obtain the following necessary conditions for cyclicity in $\mathcal{D}(\mathbb{T})$.

Theorem 1. *If f is cyclic for $\mathcal{D}(\mathbb{T})$, then*

$$\int_{\mathbb{T}} \log |f(\zeta)| |d\zeta| = -\infty \quad \text{and} \quad c(\mathcal{Z}(f)) = 0.$$

Our goal in this paper is to give sufficient conditions for a function $f \in \mathcal{D}(\mathbb{T})$ to be cyclic.

For $\beta \in (0, 1]$, we shall denote by $\text{Lip}_\beta(\mathbb{T})$ the set of functions f continuous on \mathbb{T} such that

$$\|f\|_{\text{Lip}_\beta(\mathbb{T})} := \|f\|_{C(\mathbb{T})} + \sup_{\zeta, \zeta' \in \mathbb{T}} \frac{|f(\zeta) - f(\zeta')|}{|\zeta - \zeta'|^\beta} < \infty.$$

For $\alpha \in (0, 1)$, we set

$$\mathcal{C}^{1+\alpha}(\mathbb{T}) := \{f \in C^1(\mathbb{T}) : f' \in \text{Lip}_\alpha(\mathbb{T})\}.$$

Of course, if f belongs to $\text{Lip}_\beta(\mathbb{T})$ or $\mathcal{C}^{1+\alpha}(\mathbb{T})$, then $\mathcal{Z}(f)$ is closed in \mathbb{T} .

We shall establish the following result.

Theorem 2. *Let $f \in \mathcal{D}(\mathbb{T})$ such that $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$. Suppose further that $\log |f| \notin L^1(\mathbb{T})$. Then $[f^2]_{\mathbb{N}} = \mathcal{D}_{\mathcal{Z}(f)}$.*

Combining Theorems 1 and 2, we deduce

Corollary. *Let $f \in \mathcal{D}(\mathbb{T})$ such that $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$. Then the following assertions are equivalent:*

- (1) f^2 is cyclic for $\mathcal{D}(\mathbb{T})$;
- (2) $\log |f| \notin L^1(\mathbb{T})$ and $c(\mathcal{Z}(f)) = 0$.

A closed set $E \subset \mathbb{T}$ is said to be a *Carleson set* (and we write $E \in (C)$) if

$$\int_{\mathbb{T}} \log \frac{1}{d(\zeta, E)} |d\zeta| < \infty.$$

For background information on Carleson sets, see e.g. [6, §4.4]. Note that, if $f \in \text{Lip}_\beta(\mathbb{T})$ and $\mathcal{Z}(f) \notin (C)$, then $\log |f| \notin L^1(\mathbb{T})$.

It is known that $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ if and only if $\beta > 1/2$. The inclusion $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ for $\beta > 1/2$ can easily be obtained from Douglas' formula.

We shall establish the following theorem.

Theorem 3. *Let $f \in \text{Lip}_\beta(\mathbb{T})$, where $\beta \in (\frac{1}{2}, 1]$. If $\mathcal{Z}(f) \notin (C)$, then $[f]_{\mathbb{N}} = \mathcal{D}_{\mathcal{Z}(f)}$. If furthermore $c(\mathcal{Z}(f)) = 0$, then f is cyclic for $\mathcal{D}(\mathbb{T})$.*

2. PROOF OF THEOREM 2

For the proof of Theorem 2, we shall need the following standard result.

Lemma 4. *Let $f \in \mathcal{D}(\mathbb{T})$. The following assertions are equivalent:*

- (1) $[f]_{\mathbb{N}} = [f]_{\mathbb{Z}}$;
- (2) $f \in [Sf]_{\mathbb{N}}$;
- (3) $\inf\{\|pf\|_{\mathcal{D}(\mathbb{T})} : p \in H^\infty, pf \in \mathcal{D}(\mathbb{T}) \text{ and } p(0) = 1\} = 0$.

Proof. Since S is invertible, (1) and (2) are equivalent.

If $f \in [Sf]_{\mathbb{N}}$, then there is a sequence (p_n) of polynomials such that $p_n(0) = 1$ and $\|(1 - p_n)f\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$. This proves that (2) implies (3).

Finally, suppose that (3) holds. Let $(p_n) \subset H^\infty$ be a sequence such that $p_n(0) = 1$, $p_nf \in \mathcal{D}(\mathbb{T})$ and $\|p_nf\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$. Writing $p_n = 1 - zq_n$, by [12, Proposition 3.4] we have $zq_nf \in [Sf]_{\mathbb{N}}$. Since zq_nf converges to f , it follows that $f \in [Sf]_{\mathbb{N}}$, so that (2) holds. \square

We shall also need the following result, which is a special case of a theorem due to Carleson–Jacobs–Havin–Shamoyan [2, Theorem 6.1].

Lemma 5. *Let F be an outer function on \mathbb{D} that is continuous on $\overline{\mathbb{D}}$. If $|F| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in (0, 1)$, then $F \in \text{Lip}_{(1+\alpha)/2}(\overline{\mathbb{D}})$. Furthermore, the Lipschitz constant associated to F on $\overline{\mathbb{D}}$ depends only on the Lipschitz constants and bounds for the derivatives of $|F|$ on \mathbb{T} .*

Proof of Theorem 2. Let p_ϵ be the outer function such that

$$|p_\epsilon(\zeta)| = \frac{e^{-M_\epsilon}}{|f(\zeta)| + \epsilon} \quad \text{a.e. on } \mathbb{T},$$

where the constant M_ϵ is chosen so that $p_\epsilon(0) = 1$. Thus

$$M_\epsilon = \int_{\mathbb{T}} \log\left(\frac{1}{|f(\zeta)| + \epsilon}\right) \frac{|d\zeta|}{2\pi},$$

and since $\log|f| \notin L^1(\mathbb{T})$, it follows that $M_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0^+$. We are going to prove that

$$\lim_{\epsilon \rightarrow 0^+} \|p_\epsilon f^2\|_{\mathcal{D}(\mathbb{T})} = 0.$$

If this holds, then by Lemma 4 we have $[f^2]_{\mathbb{N}} = [f^2]_{\mathbb{Z}}$, and since clearly $\mathcal{Z}(f^2) = \mathcal{Z}(f)$, we can apply the Richter–Ross–Sundberg theorem to obtain the desired result.

We have

$$\|p_\epsilon f^2\|_{\mathcal{D}(\mathbb{T})}^2 = \|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(p_\epsilon f^2).$$

For the first term, we have

$$\|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} \frac{e^{-2M_\epsilon} |f|^4 |d\zeta|}{(|f| + \epsilon)^2 2\pi} \leq e^{-2M_\epsilon} \|f^2\|_{L^2(\mathbb{T})}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+.$$

The second term we estimate using Douglas' formula, namely

$$\mathcal{D}(p_\epsilon f^2) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Let

$$\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : |f(\zeta')| \leq |f(\zeta)|\}.$$

Then, by symmetry,

$$\mathcal{D}(p_\epsilon f) = 2 \frac{1}{4\pi^2} \iint_{\Gamma} \frac{|(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Now, for all $\zeta, \zeta' \in \mathbb{T}$, we have

$$\begin{aligned} & |(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2 \\ &= |p_\epsilon(\zeta)(f^2(\zeta) - f^2(\zeta')) + f^2(\zeta')(p_\epsilon(\zeta) - p_\epsilon(\zeta'))|^2 \\ &\leq 2|p_\epsilon(\zeta)|^2 |f^2(\zeta) - f^2(\zeta')|^2 + 2|f^2(\zeta')|^2 |p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2. \end{aligned}$$

Hence

$$\iint_{\Gamma} \frac{|(p_{\epsilon}f^2)(\zeta) - (p_{\epsilon}f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \leq 2A_{\epsilon} + 2B_{\epsilon},$$

where

$$A_{\epsilon} := \iint_{\Gamma} |p_{\epsilon}(\zeta)|^2 \frac{|f^2(\zeta) - f^2(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|$$

and

$$B_{\epsilon} := \iint_{\Gamma} |f^2(\zeta')|^2 \frac{|p_{\epsilon}(\zeta) - p_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

We estimate A_{ϵ} directly as follows:

$$\begin{aligned} A_{\epsilon} &= e^{-2M_{\epsilon}} \iint_{\Gamma} \frac{|f(\zeta) + f(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq 4e^{-2M_{\epsilon}} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq 4e^{-2M_{\epsilon}} 4\pi^2 \mathcal{D}(f). \end{aligned}$$

Hence $A_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^+$.

To estimate B_{ϵ} , we consider the outer function F_{ϵ} such that

$$|F_{\epsilon}(\zeta)| = |f(\zeta)| + \epsilon \quad \text{a.e. on } \mathbb{T}.$$

By Lemma 5, since $|F_{\epsilon}| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, we have $F_{\epsilon} \in \text{Lip}_{(1+\alpha)/2}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ and there exists a positive constant D , depending only on $|f|$, such that $\mathcal{D}(F_{\epsilon}) \leq D$ for all $\epsilon \in (0, 1)$. We then have

$$\begin{aligned} B_{\epsilon} &= \iint_{\Gamma} \frac{e^{-2M_{\epsilon}} |f^2(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2 (|f(\zeta')| + \epsilon)^2} \frac{|1/p_{\epsilon}(\zeta) - 1/p_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq e^{-2M_{\epsilon}} \iint_{\Gamma} \frac{|F_{\epsilon}(\zeta) - F_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq e^{-2M_{\epsilon}} 4\pi^2 \mathcal{D}(F_{\epsilon}). \end{aligned}$$

Thus $B_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^+$. This completes the proof of Theorem 2. \square

3. PROOF OF THEOREM 3

To prove Theorem 3, we shall need the following additional lemma.

Lemma 6. *Let $f \in \text{Lip}_{\beta}(\mathbb{T})$, where $\beta > 1/2$. Then, for $\eta \in (0, \frac{2\beta-1}{2\beta})$, we have*

$$\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| < +\infty.$$

Proof. Since $\beta > 1/2(1 - \eta)$, we get

$$\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \lesssim \iint_{\mathbb{T}^2} \frac{|d\zeta'| |d\zeta|}{|\zeta - \zeta'|^{2(1-(1-\eta)\beta)}} < \infty. \quad \square$$

Note that here, and in what follows, we write $A \lesssim B$ to mean that there is an absolute constant C such that $A \leq CB$.

Proof of Theorem 3. By [12], it suffices to prove that $[f]_{\mathbb{N}}$ is 2-invariant, which is equivalent to proving that $f \in [Sf]_{\mathbb{N}}$.

Let $\epsilon, \gamma > 0$, where γ will be taken small. Let E be a closed subset of $\mathcal{Z}(f)$ such that $|E| = 0$ and $E \notin (C)$. Let p_ϵ be the outer function satisfying

$$|p_\epsilon(\zeta)| = \frac{e^{-M_\epsilon}}{(d(\zeta, E)^\gamma + \epsilon)^{1/2}} \quad \text{a.e. on } \mathbb{T},$$

where the constant M_ϵ is chosen so that $p_\epsilon(0) = 1$. Thus

$$M_\epsilon := \frac{1}{2} \int_{\mathbb{T}} \log \left(\frac{1}{d(\zeta, E)^\gamma + \epsilon} \right) \frac{|d\zeta|}{2\pi},$$

and since $E \notin (C)$, it follows that $M_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0^+$. By Lemma 4, it suffices to prove that

$$\lim_{\epsilon \rightarrow 0^+} \|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})} = 0.$$

Now

$$\|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})}^2 = \|p_\epsilon f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(p_\epsilon f).$$

For the first term, we have

$$\|p_\epsilon f\|_{L^2(\mathbb{T})}^2 \lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{d(\zeta, E)^{2\beta}}{d(\zeta, E)^\gamma} |d\zeta|.$$

Thus $\|p_\epsilon f\|_{L^2(\mathbb{T})} \rightarrow 0$ as $\epsilon \rightarrow 0^+$, provided that $\gamma < 2\beta$.

For the second term we again use Douglas' formula, namely

$$\mathcal{D}(p_\epsilon f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Let

$$\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : d(\zeta', E) \leq d(\zeta, E)\}.$$

Arguing as in the proof of Theorem 2, we have

$$\begin{aligned} \mathcal{D}(p_\epsilon f) &\lesssim \iint_{\Gamma} |p_\epsilon(\zeta)|^2 \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &\quad + \iint_{\Gamma} |f(\zeta')|^2 \frac{|p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &= A_\epsilon + B_\epsilon, \text{ say.} \end{aligned}$$

To estimate A_ϵ , let $\eta \in (0, (2\beta - 1)/(2\beta))$ and let $\gamma < 2\beta\eta$. Then

$$\begin{aligned} A_\epsilon &= e^{-2M_\epsilon} \iint_{\Gamma} \left(\frac{|f(\zeta) - f(\zeta')|^{2\eta}}{d(\zeta, E)^\gamma + \epsilon} \right) \left(\frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} \right) |d\zeta| |d\zeta'| \\ &\leq 2C e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{d(\zeta, E)^{2\beta\eta}}{d(\zeta, E)^\gamma} \left(\frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} \right) |d\zeta| |d\zeta'| \\ &\leq 2C e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'|, \end{aligned}$$

where C depends only on f , and the double integral is finite, thanks to Lemma 6. Hence $A_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0^+$.

To estimate B_ϵ , we introduce the outer function F_ϵ satisfying

$$|F_\epsilon(\zeta)| = d(\zeta, E)^\gamma + \epsilon \quad \text{a.e. on } \mathbb{T}.$$

By the Carleson–Richter–Sundberg formula [6, Theorem 7.4.2], we have

$$\mathcal{D}_\zeta(F_\epsilon) = \int_{\mathbb{T}} \frac{|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2 - 2|F_\epsilon(\zeta')| \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \frac{|d\zeta'|}{2\pi}.$$

Therefore

$$\begin{aligned} B_\epsilon &\lesssim e^{-2M_\epsilon} \iint_{\Gamma} \frac{d(\zeta', E)^{2\beta}}{(d(\zeta, E)^\gamma + \epsilon)(d(\zeta', E)^\gamma + \epsilon)} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} d(\zeta', E)^{2(\beta-\gamma)} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \mathcal{D}_\zeta(F_\epsilon) d(\zeta, E)^{2(\beta-\gamma)} |d\zeta| \\ &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \left(\frac{|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2 - 2|F_\epsilon(\zeta')| \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \right) \\ &\quad \times \left(d(\zeta, E)^{2(\beta-\gamma)} + d(\zeta', E)^{2(\beta-\gamma)} \right) |d\zeta| |d\zeta'|. \end{aligned}$$

Exchanging the roles of ζ and ζ' , and taking the average, we obtain

$$\begin{aligned} B_\epsilon &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \left(\frac{(|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2) \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \right) \\ &\quad \times \left(d(\zeta, E)^{2(\beta-\gamma)} + d(\zeta', E)^{2(\beta-\gamma)} \right) |d\zeta| |d\zeta'|. \end{aligned}$$

Thus

$$B_\epsilon \lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{\delta^\gamma - \delta'^\gamma}{|\zeta - \zeta'|^2} \log \left(\frac{\delta^\gamma + \epsilon}{\delta'^\gamma + \epsilon} \right) (\delta^{2(\beta-\gamma)} + \delta'^{2(\beta-\gamma)}) |d\zeta| |d\zeta'|, \quad (3.1)$$

where $\delta := d(\zeta, E)$ and $\delta' := d(\zeta', E)$.

Let (I_j) be the connected components of $\mathbb{T} \setminus E$, and set

$$N_E(t) := 2 \sum_j 1_{\{|I_j| > 2t\}}, \quad 0 < t < 1.$$

Then, for every measurable function $\Omega : [0, \pi] \rightarrow \mathbb{R}^+$, we have

$$\int_{\mathbb{T}} \Omega(d(\zeta, E)) |d\zeta| = \int_0^\pi \Omega(t) N_E(t) dt.$$

Using similar ideas to those in [4, 5], we obtain

$$\begin{aligned} J &:= \iint_{\mathbb{T}^2} \frac{\delta^\gamma - \delta'^\gamma}{|\zeta - \zeta'|^2} \log\left(\frac{\delta^\gamma + \epsilon}{\delta'^\gamma + \epsilon}\right) (\delta^{2(\beta-\gamma)} + \delta'^{2(\beta-\gamma)}) |d\zeta| |d\zeta'| \\ &\lesssim \int_0^\pi \int_0^\pi \frac{((s+t)^\gamma - t^\gamma)}{s^2} \log\left(\frac{(s+t)^\gamma + \epsilon}{t^\gamma + \epsilon}\right) (t+s)^{2(\beta-\gamma)} N_E(t) ds dt \\ &\lesssim \int_0^\pi \int_0^t \frac{((s+t)^\gamma - t^\gamma) \log[(s+t)^\gamma/t^\gamma]}{s^2} (t+s)^{2(\beta-\gamma)} N_E(t) ds dt \\ &\quad + \int_0^\pi \int_t^\pi \frac{((s+t)^\gamma - t^\gamma)}{s^2} \log[1/(t^\gamma + \epsilon)] (t+s)^{2(\beta-\gamma)} ds N_E(t) dt \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &\lesssim \int_0^\pi t^{2\beta-\gamma-1} \int_0^1 \frac{((1+x)^\gamma - 1)}{x^2} \log(1+x) dx N_E(t) dt \\ &\lesssim \int_0^\pi t^{2\beta-\gamma-1} N_E(t) dt = O(1), \end{aligned}$$

and

$$\begin{aligned} J_2 &\lesssim \int_0^\pi t^{2\beta-\gamma-1} \log[1/(t^\gamma + \epsilon)] \int_1^{\pi/t} \frac{(1+x)^\gamma - 1}{s^2} (1+x)^{2(\beta-\gamma)} dx N_E(t) dt \\ &\lesssim \int_0^\pi \log[1/(t^\gamma + \epsilon)] N_E(t) dt \\ &\lesssim \int_{\mathbb{T}} |\log(d(\zeta, E)^\gamma + \epsilon)| |d\zeta| = 2M_\epsilon. \end{aligned}$$

Thus $J = O(M_\epsilon)$. Combining this with the estimate (3.1), we get

$$B_\epsilon \lesssim M_\epsilon e^{-2M_\epsilon}.$$

Hence $B_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0^+$. This completes the proof of Theorem 3. \square

4. CONCLUDING REMARKS

1. In order to produce cyclic functions for $\mathcal{D}(\mathbb{T})$ using Theorem 3, we need to construct closed subsets $E \subset \mathbb{T}$ such that $E \notin (C)$ and $c(E) = 0$. An easy example can be given by countable sets. Indeed, taking $E_\beta := \{e^{i/(\log n)^\beta} : n \geq 2\}$ with $\beta \leq 1$ provides such an example. Using Cantor-type sets, it is also possible to construct perfect sets E such that $E \notin (C)$ and $c(E) = 0$.

2. One can consider weighted harmonic Dirichlet spaces instead of the classical harmonic Dirichlet space. More precisely, given $\alpha \in [0, 1)$, the *weighted harmonic Dirichlet space* $\mathcal{D}_\alpha(\mathbb{T})$ is the space of functions $f \in L^2(\mathbb{T})$ such that

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 (1 + |n|)^{1-\alpha} < \infty.$$

We define the α -*capacity* of a Borel subset $E \subset \mathbb{T}$ by

$$c_\alpha(E) = 1/\inf\{I_\alpha(\mu) : \mu \in \mathcal{P}(E)\},$$

where $\mathcal{P}(E)$ is the set of all probability measures supported on a compact subset of E and $I_\alpha(\mu) := \sum_{n \geq 1} |\hat{\mu}(n)|^2/n^{1-\alpha}$ is the α -*energy* of μ . We say that a property holds c_α -*quasi-everywhere* if it holds everywhere outside a set of c_α -capacity zero.

It is well known that $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}_\alpha(\mathbb{T})$ if and only $\beta > (1 - \alpha)/2$. Theorem 3 may be extended to show that, if $f \in \text{Lip}_\beta(\mathbb{T})$, where $\beta \in ((1 - \alpha)/2, 1]$, and if $\mathcal{Z}(f) \notin (C)$, then

$$[f]_{\mathbb{N}} = \{g \in \mathcal{D}(\mathbb{T}) : g|_{\mathcal{Z}(f)} = 0 \text{ } c_\alpha\text{-quasi-everywhere}\}.$$

3. One can equally well consider the holomorphic Dirichlet space, namely $\mathcal{D} := \{f \in \mathcal{D}(\mathbb{T}) : \hat{f}(n) = 0 \text{ } (n < 0)\}$. Here too the problem of characterizing the cyclic functions is still open. For more on this topic, see e.g. [6, Chapter 9].

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